

# Finite volume HWENO schemes for nonconvex conservation laws\*

Xiaofeng Cai<sup>†</sup>, Jianxian Qiu<sup>‡</sup> and Jingmei Qiu<sup>§</sup>

**Abstract:** We illustrate that numerical solutions of high order finite volume Hermite weighted essentially non-oscillatory (HWENO) scheme for some nonconvex conservation laws perform poorly or converge to the entropy solution in a slow speed. The modified finite volume HWENO schemes based either on first order monotone schemes or a second order entropic projection following the work of Qiu and Shu [SIAM J. Sci. Comput., 31 (2008), 584-607] are proposed and compared for solving one-dimensional scalar problems. We extend the modified finite volume HWENO based on first order monotone schemes for one-dimensional systems and two-dimensional scalar conservation laws. Numerical tests for several representative examples will be reported.

**Keywords:** nonconvex conservation laws; finite volume HWENO scheme; entropy solution; entropic projection.

---

\*Research was supported by NSFC grants 11571290, 91530107 and NSF DMS-1217008 and DMS-1522777.

<sup>†</sup>School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling & High-Performance Scientific Computing, Xiamen University, Xiamen, Fujian, 361005, P.R. China. E-mail: xfcai@stu.xmu.edu.cn.

<sup>‡</sup>School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling & High-Performance Scientific Computing, Xiamen University, Xiamen, Fujian, 361005, P.R. China. E-mail: jxqiu@xmu.edu.cn.

<sup>§</sup>Department of Mathematics, University of Houston, Houston, 77204. E-mail: jingqiu@math.uh.edu.

# 1 Introduction

In this paper, we consider the Cauchy problem for nonconvex hyperbolic conservation laws:

$$\begin{cases} q_t + \nabla \cdot F(q) = 0, \\ q(\mathbf{x}, 0) = q_0(\mathbf{x}), \end{cases} \quad (1.1)$$

whose entropy solutions may admit composite waves which involve a sequence of shocks and rarefaction waves and are difficult to be resolved numerically. Such examples include scalar conservation laws with nonconvex flux functions and hyperbolic systems such as the Euler system and magnetohydrodynamics system with a nonconvex equation of state (EOS) [6, 19, 7, 12].

It is well known that first order monotone schemes converge to entropy solutions of both convex and nonconvex conservation laws [3], but with a relatively slow convergence rate. In [4], a high order entropy viscosity method was proposed for both convex and nonconvex conservation laws. There are examples of nonconvex conservation laws [21], for which high order Godunov schemes (e.g., finite volume (FV) weighted essentially non-oscillatory (WENO) schemes [13, 14] and discontinuous Galerkin methods [2, 22]) would fail to converge to the entropy solution. An adaptive strategy combining a low order dissipative method with a high order method was proposed in [5], for which the dissipative reconstruction is used to reconstruct the information near the region where the flux convexity changes. Recently high order FV WENO schemes and discontinuous Galerkin methods with a first order monotone modification, as well as a second order MUSCL type scheme satisfying all entropy inequalities are proposed in [11]. The proposed modifications enforce convergence towards the entropy solution of nonconvex conservation laws.

The FV HWENO scheme was proposed in [8, 9, 24, 17]. There are quite a few advantages of HWENO using a relatively compact numerical stencil: first, it is easier to deal with boundary conditions and complex geometries; second, for the same formal accuracy, compact stencils are known to exhibit more resolution of the smaller scales by improving the dispersive and the dissipative properties of the numerical scheme [18]. In this paper, we follow the idea in [11] and propose the FV HWENO scheme with a first order monotone modification and a second order MUSCL type scheme satisfying all entropy inequalities to enforce entropy convergence. The modified FV HWENO scheme may generate oscillations for some challenging problems, for which we apply the monotonicity preserving (MP) limiter of Suresh and Huynh [16] to the HWENO scheme. We also design the FV HWENO scheme with a first order monotone modification for 1D systems and 2D problems.

An outline of this paper is as follows. Section 2 describes the high order FV HWENO scheme. In Section 3,

FV HWENO schemes with a first order monotone modification and a second order modification with an entropic projection will be proposed for 1D nonconvex conservation laws and we extend the scheme to 2D problems. In Section 4, numerical examples are shown to demonstrate the quality of proposed schemes. Concluding remarks are given in Section 5.

## 2 Description of FV HWENO schemes

We briefly review the FV HWENO schemes for solving conservation laws [8, 9, 24]. The idea of HWENO method is to numerically evolve *both the function and its spatial gradients*, and use these information in the reconstruction process, thus leading to a more compact reconstruction stencil compared with the regular WENO scheme [13, 14].

Taking the gradient with respect to the spatial variables in (1.1), we have

$$(\nabla q)_t^T + \nabla \cdot (\nabla \otimes F(q)) = 0, \quad (2.1)$$

where  $\otimes$  is a tensor product. For using a Hermite interpolation procedure, both the function and its gradients are needed during the evolution in time. The FV HWENO schemes are defined for the equations:

$$\mathbf{U}_t + \nabla \cdot \mathcal{F}(\mathbf{U}) = 0, \quad (2.2)$$

where  $\mathbf{U} = (q, \nabla q)^T$  and  $\mathcal{F}(\mathbf{U}) = \begin{pmatrix} F(q) \\ \nabla \otimes F(q) \end{pmatrix}$ . We integrate the system (2.2) on a control volume  $\Omega_k$ , which is an interval  $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$  for 1D cases or a rectangle  $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$  for 2D cases. The integral form of (2.2) reads,

$$\frac{d}{dt} \overline{\mathbf{U}}_{\Omega_k} = - \frac{1}{|\Omega_k|} \int_{\partial\Omega_k} \mathcal{F}(\mathbf{U}) \cdot \mathbf{n} ds \quad (2.3)$$

where  $|\Omega_k|$  is the volume of  $\Omega_k$  and  $\mathbf{n}$  represents the outward unit normal vector to  $\partial\Omega_k$ . The line integral in (2.3) can be approximated by a  $L$ -point Gaussian quadrature on each side of  $\partial\Omega_k = \bigcup_{s=1}^S \partial\Omega_{ks}$ :

$$\int_{\partial\Omega_k} \mathcal{F}(\mathbf{U}) \cdot \mathbf{n} ds \approx \sum_{s=1}^S |\partial\Omega_{ks}| \sum_{l=1}^L \omega_l \mathcal{F}(\mathbf{U}(G_{sl}, t)) \cdot \mathbf{n}, \quad (2.4)$$

where  $G_{sl}$  and  $\omega_l$  are Gaussian quadrature points on  $\partial\Omega_{ks}$  and weights respectively.  $\mathcal{F}(\mathbf{U}(G_{sl}, t)) \cdot \mathbf{n}$  is evaluated by a numerical flux (approximate or exact Riemann solvers). We adopt the Lax-Friedrichs flux in this paper, which is given by

$$\mathcal{F}(\mathbf{U}(G_{sl}, t)) \cdot \mathbf{n} \approx \frac{1}{2} [\mathcal{F}(\mathbf{U}^-(G_{sl}, t)) + \mathcal{F}(\mathbf{U}^+(G_{sl}, t))] \cdot \mathbf{n} - \alpha (\mathbf{U}^+(G_{sl}, t) - \mathbf{U}^-(G_{sl}, t)),$$

where  $\alpha$  is taken as an upper bound for the eigenvalues of the Jacobian along the direction  $\mathbf{n}$ , and  $\mathbf{U}^-$  and  $\mathbf{U}^+$  are the reconstructed values of  $\mathbf{U}$  at the Gaussian point  $G_{sl}$  inside and outside  $\Omega_k$ . The procedures of FV HWENO reconstruction of  $\mathbf{U}^\pm(G_{sl}, t)$  for 1D cases and 2D cases are given in details in [8, 24]. Finally, the semi-discretization HWENO scheme (2.3) can be written in the following ODE form:

$$\frac{d}{dt} \bar{\mathbf{U}} = \mathcal{L}(\bar{\mathbf{U}}). \quad (2.5)$$

The ODE system (2.5) is then discretized in time by a SSP Runge-Kutta method in [15]. The third-order version is used in this paper,

$$\begin{aligned} \bar{\mathbf{U}}^{(1)} &= \bar{\mathbf{U}}^n + \Delta t \mathcal{L}(\bar{\mathbf{U}}^n), \\ \bar{\mathbf{U}}^{(2)} &= \frac{3}{4} \bar{\mathbf{U}}^n + \frac{1}{4} (\bar{\mathbf{U}}^{(1)} + \Delta t \mathcal{L}(\bar{\mathbf{U}}^{(1)})), \\ \bar{\mathbf{U}}^{n+1} &= \frac{1}{3} \bar{\mathbf{U}}^n + \frac{2}{3} (\bar{\mathbf{U}}^{(2)} + \Delta t \mathcal{L}(\bar{\mathbf{U}}^{(2)})). \end{aligned} \quad (2.6)$$

### 3 The modified FV HWENO schemes for nonconvex conservation laws

Although FV HWENO schemes can be successfully applied in many applications [8, 9, 24, 10, 23], they could perform poorly for nonconvex conservation laws as demonstrated below. To remedy this, we propose a first order monotone modification and a second order modification with an entropic projection for the FV HWENO scheme.

#### 3.1 The example of nonconvex conservation laws with poor performance for FV HWENO schemes

In the subsection, we will show a nonconvex conservation law, for which the FV HWENO scheme performs poorly in converging to the entropy solution.

Consider the FV HWENO scheme for solving

$$q_t + f(q)_x = 0 \quad (3.1)$$

with the nonconvex flux  $f(q) = \sin(q)$  and the initial condition

$$q_0(x) = \begin{cases} \pi/64, & \text{if } x < 0, \\ 255\pi/64, & \text{if } x \geq 0. \end{cases} \quad (3.2)$$

It is shown, in Figure 3.1, that the numerical solution of the high order FV HWENO scheme does not converge to the entropy solution which is given by the first order Godunov scheme with a very refined mesh. One of the rarefaction waves in the compound wave is missing.

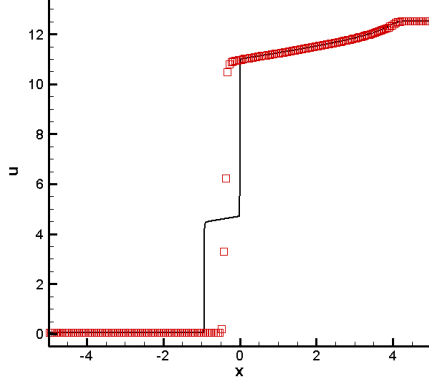


Figure 3.1: Solid lines: the reference solution of (3.1) with  $f(q) = \sin(q)$  and the initial condition (4.3) at the time  $t = 4$ ; Squares: FV HWENO scheme with Lax-Friedrichs flux. The mesh is uniform with  $\Delta x = 0.05$ .

### 3.2 First order monotone modification

In this subsection, we propose a first order modification to the FV HWENO scheme for 1D nonconvex conservation laws.

Let  $\xi = q_x$ . Taking the derivative of (3.1), we have

$$\xi_t + \mathcal{H}(q, \xi)_x = 0, \quad (3.3)$$

where  $\mathcal{H}(q, \xi) = f'(q)\xi$ . Let  $\bar{q}_j$  and  $\bar{\xi}_j$  denote approximation to the cell average of  $q$  and  $\xi$  over cell  $I_j$  respectively.

The FV HWENO scheme is designed by integrating equation (3.1) and (3.3),

$$\begin{cases} \frac{d\bar{q}_j}{dt} = -\frac{\hat{f}\left(q_{j+\frac{1}{2}}^-, q_{j+\frac{1}{2}}^+\right) - \hat{f}\left(q_{j-\frac{1}{2}}^-, q_{j-\frac{1}{2}}^+\right)}{\Delta x}, \\ \frac{d\bar{\xi}_j}{dt} = -\frac{\hat{\mathcal{H}}\left(q_{j+\frac{1}{2}}^-, q_{j+\frac{1}{2}}^+; \xi_{j+\frac{1}{2}}^-, \xi_{j+\frac{1}{2}}^+\right) - \hat{\mathcal{H}}\left(q_{j-\frac{1}{2}}^-, q_{j-\frac{1}{2}}^+; \xi_{j-\frac{1}{2}}^-, \xi_{j-\frac{1}{2}}^+\right)}{\Delta x}, \end{cases} \quad (3.4)$$

where  $\hat{f}(a, b)$  is a monotone numerical flux (non-decreasing in the first argument and non-increasing in the second argument).  $\hat{\mathcal{H}}(a, b; c, d)$  is non-decreasing in the third argument and non-increasing in the fourth argument. In this paper, we use the Lax-Friedrichs flux in [8].

$$\begin{aligned} \hat{f}(a, b) &= \frac{1}{2}[f(a) + f(b) - \alpha(b - a)], \\ \hat{\mathcal{H}}(a, b; c, d) &= \frac{1}{2}[\mathcal{H}(a, c) + \mathcal{H}(b, d) - \alpha(d - c)], \end{aligned} \quad (3.5)$$

where  $\alpha = \max_q |f'(q)|$ . Total variation stability of this Lax-Friedrichs scheme is proved in [8].

The scheme can be summarized as follows, after a suitable initialization to obtain  $\bar{q}^0$  and  $\bar{\xi}^0$ .

1. Perform the HWENO reconstruction [8].

At each cell interface, say  $x_{j+\frac{1}{2}}$ , reconstruct the point values  $q_{j+\frac{1}{2}}^\pm$  and derivative values  $\xi_{j+\frac{1}{2}}^\pm$  using neighboring cell average  $\bar{q}$  and  $\bar{\xi}$  respectively by the fifth order HWENO reconstruction procedure in Section 2.

2. Identify the troubled cell boundary  $x_{j+\frac{1}{2}}$ .

*Criterion I: A cell boundary  $x_{j+\frac{1}{2}}$  is good, if  $q_{j+\frac{1}{2}}^\pm$ ,  $\bar{q}_j$  and  $\bar{q}_{j+1}$  all fall into the same linear, convex or concave region of the flux function  $f(q)$ . Otherwise, it is defined to be a troubled cell boundary.*

3. At troubled cell boundaries, modify the numerical flux  $\hat{f}_{j+\frac{1}{2}}$  and  $\hat{\mathcal{H}}_{j+\frac{1}{2}}$  with a discontinuity indicator in [20]. Specifically, the discontinuity indicator  $\phi_j$  is defined as

$$\phi_j = \frac{\beta_j}{\beta_j + \gamma_j} \quad (3.6)$$

where

$$\alpha_j = |\bar{q}_{j-1} - \bar{q}_j|^2 + \varepsilon, \quad \tau_j = |\bar{q}_{j+1} - \bar{q}_{j-1}|^2 + \varepsilon, \quad \beta_j = \frac{\tau_j}{\alpha_{j-1}} + \frac{\tau_j}{\alpha_{j+2}}, \quad \gamma_j = \frac{(q_{max} - q_{min})^2}{\alpha_j}.$$

Here  $\varepsilon$  is a small positive number taken as  $10^{-6}$  in the code, and  $q_{max}$  and  $q_{min}$  are the maximum and minimum values of  $\bar{q}_j$  over all cells. The discontinuity indicator  $\phi_j$  has the property that

- $0 \leq \phi_j \leq 1$ .
- $\phi_j$  is on the order of  $O(\Delta x^2)$  in smooth regions.
- $\phi_j$  is close to  $O(1)$  near a strong discontinuity.

Let  $\hat{f}_{j+\frac{1}{2}} = \hat{f}(q_{j+\frac{1}{2}}^{m,-}, q_{j+\frac{1}{2}}^{m,+})$ , and  $\hat{\mathcal{H}}_{j+\frac{1}{2}} = \hat{\mathcal{H}}(q_{j+\frac{1}{2}}^{m,-}, q_{j+\frac{1}{2}}^{m,+}; \xi_{j+\frac{1}{2}}^{m,-}, \xi_{j+\frac{1}{2}}^{m,+})$  where

$$q_{j+\frac{1}{2}}^{m,-} = (1 - \phi_j^2)q_{j+\frac{1}{2}}^- + \phi_j^2\bar{q}_j, \quad q_{j+\frac{1}{2}}^{m,+} = (1 - \phi_j^2)q_{j+\frac{1}{2}}^+ + \phi_j^2\bar{q}_{j+1},$$

$$\xi_{j+\frac{1}{2}}^{m,-} = (1 - \phi_j^2)\xi_{j+\frac{1}{2}}^- + \phi_j^2\bar{\xi}_j, \quad \xi_{j+\frac{1}{2}}^{m,+} = (1 - \phi_j^2)\xi_{j+\frac{1}{2}}^+ + \phi_j^2\bar{\xi}_{j+1},$$

with  $\phi_j$  defined by (3.6), if  $x_{j+\frac{1}{2}}$  is a troubled cell boundary. Otherwise, at good cell boundaries,  $q_{j+\frac{1}{2}}^{m,\pm} = q_{j+\frac{1}{2}}^\pm$  and  $\xi_{j+\frac{1}{2}}^{m,\pm} = \xi_{j+\frac{1}{2}}^\pm$ .

4. Evolve the cell averages  $\bar{q}_j$  and  $\bar{\xi}_j$  by (3.4).

**REMARK 1.** When a troubled cell boundary is at a strong discontinuity,  $\phi_j \sim 1$ , hence  $q_{j+\frac{1}{2}}^{m,-} \sim \bar{q}_j$ ,  $q_{j+\frac{1}{2}}^{m,+} \sim \bar{q}_{j+1}$ ,  $\xi_{j+\frac{1}{2}}^{m,-} \sim \bar{\xi}_j$  and  $\xi_{j+\frac{1}{2}}^{m,+} \sim \bar{\xi}_{j+1}$ , indicating a first order monotone scheme is taking the effect at a nonconvex

discontinuous region. When a troubled cell boundary is in a smooth region, the modification is obtained with the magnitude at most of the size

$$\phi_j^2 \max(|q_j - q_{j+\frac{1}{2}}^-|, |q_{j+1} - q_{j+\frac{1}{2}}^+|) \sim O(\Delta x^5),$$

$$\phi_j^2 \max(|\xi_j - \xi_{j+\frac{1}{2}}^-|, |\xi_{j+1} - \xi_{j+\frac{1}{2}}^+|) \sim O(\Delta x^5),$$

hence it does not affect the fifth order accuracy of the scheme.

### 3.3 Second order modification with an entropic projection

A MUSCL type method with an entropic projection which enjoy cell entropy inequalities for all convex entropy functions, is proposed in [1]. A second order modification for the high order FV HWENO schemes will be constructed following the similar procedure in [11].

#### 3.3.1 Review of the MUSCL method satisfying all the numerical entropy inequalities

Piecewise linear functions as the solution space is used in the scheme in [1]. Specifically, the numerical solution at time level  $n$  can be written as  $u^n = \bar{u}_j^n + s_j^n \sigma_j$  with  $\sigma_j = \frac{x-x_j}{\Delta x_j}$  over the cell  $I_j$ . It consists of two steps to evolve from  $q^n$  to  $q^{n+1}$ .

1. *Exact evolution* ( $T_{\Delta x}$ ): Evolve (3.1) exactly for a time step  $\Delta t$ , to obtain a solution  $\tilde{q}^{n+1}$ , which in general is not a piecewise linear function anymore.
2. *An entropic projection* ( $P^1$ ): Find a second order approximation to  $\tilde{q}^{n+1}$  by a piecewise linear function  $q^{n+1}$ , satisfying

$$\int_{I_j} U(q^{n+1}(x))dx \leq \int_{I_j} U(\tilde{q}^{n+1}(x))dx, \quad \forall j \quad (3.7)$$

for all convex entropy function  $U(u)$ . Second order reconstruction satisfying (3.7) can be obtain by setting the cell average as

$$\bar{q}_j^{n+1} = \frac{1}{\Delta x_j} \int_{I_j} \tilde{q}^{n+1} \quad (3.8)$$

and the slope as

$$s_j^{n+1} = D\tilde{q}^{n+1}|_{I_j} = \min\text{mod}_{I_j} \zeta(y) \quad (3.9)$$

where

$$\zeta(y) = \frac{2}{\Delta x_j} \left( \frac{1}{x_{j+\frac{1}{2}} - y} \int_y^{x_{j+\frac{1}{2}}} \tilde{q}^{n+1}(x)dx - \frac{1}{y - x_{j-\frac{1}{2}}} \int_{x_{j-\frac{1}{2}}}^y \tilde{q}^{n+1}(x)dx \right) \quad (3.10)$$

The minmod function of  $g(x)$  on the interval  $(a, b)$  is defined as

$$\text{minmod}_{(a,b)} g(x) = \begin{cases} 0, & \text{if } \exists y_1, y_2 \in (a, b), \text{ s.t. } g(y_1)g(y_2) \leq 0, \\ \min_{(a,b)} g(y), & \text{if } g(y) > 0, \forall y \in (a, b), \\ \max_{(a,b)} g(y), & \text{if } g(y) < 0, \forall y \in (a, b). \end{cases} \quad (3.11)$$

In summary, the scheme can be written out in the following abstract form

$$q^{n+1} = P^1 \circ T_{\Delta t}(q^n) \doteq Q^1(\Delta t)(q^n). \quad (3.12)$$

It enjoys the following convergence theorem as proved in [1].

**THEOREM 1.** [1]. *Let  $T = n\Delta t$ ,  $u(\cdot, T)$  be the exact entropy solution to (3.1) with the initial data  $q_0 \in L^1 \cap BV(\mathcal{R})$ ,  $f_\infty = \max_{q \in [\min q_0, \max q_0]} f'(q)$ , then there exists a constant  $C$ , such that*

$$\|Q^1(\Delta t)^n q^0 - q(\cdot, T)\|_{L^1} \leq C(f_\infty \sqrt{T\Delta t} + \Delta x \sqrt{T/\Delta t}). \quad (3.13)$$

Therefore, the second order MUSCL scheme with the entropic projection (3.12) converges to the unique entropy solution.

### 3.4 Second order modification to the fifth order FV HWENO schemes

At each time step evolution,  $\{\bar{q}_j, \bar{\xi}_j, q_{j \mp \frac{1}{2}}^\pm, q_j^l, q_j^r\}$  over the cell  $I_j$  is updated. At the initial stage,  $q_{j \mp \frac{1}{2}}^\pm$  is obtain by the HWENO reconstruction from  $\bar{q}$  and  $\bar{\xi}$ .  $q_j^l$  and  $q_j^r$  refer to approximations to the left and right boundaries of  $I_j$ .  $q_j^l = q_{j-\frac{1}{2}}^+$  and  $q_j^r = q_{j+\frac{1}{2}}^-$ .

1. Identify the troubled cell boundaries, for which we refer to Criterion I in section 3.2 for the details.
2. Modify the numerical flux  $\hat{f}_{j+\frac{1}{2}}$  and  $\hat{\mathcal{H}}_{j+\frac{1}{2}}$  with a discontinuity indicator.

Specifically, let  $\hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j+\frac{1}{2}}^{m,-}, u_{j+\frac{1}{2}}^{m,+})$  and  $\hat{\mathcal{H}}_{j+\frac{1}{2}} = \hat{\mathcal{H}}(u_{j+\frac{1}{2}}^{m,-}, u_{j+\frac{1}{2}}^{m,+}; \xi_{j+\frac{1}{2}}^{m,-}, \xi_{j+\frac{1}{2}}^{m,+})$ , where

$$q_{j+\frac{1}{2}}^{m,-} = (1 - \phi_j^2) q_{j+\frac{1}{2}}^- + \phi_j^2 q_j^r, \quad q_{j+\frac{1}{2}}^{m,+} = (1 - \phi_j^2) q_{j+\frac{1}{2}}^+ + \phi_j^2 q_{j+1}^l, \quad (3.14)$$

$$\xi_{j+\frac{1}{2}}^{m,-} = (1 - \phi_j^2) \xi_{j+\frac{1}{2}}^- + \phi_j^2 \bar{\xi}_j, \quad \xi_{j+\frac{1}{2}}^{m,+} = (1 - \phi_j^2) \xi_{j+\frac{1}{2}}^+ + \phi_j^2 \bar{\xi}_{j+1}, \quad (3.15)$$

with  $\phi_j$  defined by (3.6) at the troubled cell boundary. At good cell boundaries,  $q_{j+\frac{1}{2}}^{m,\pm} = q_{j+\frac{1}{2}}^\pm$ .

3. Identify nonconvex troubled cells.

*Criterion II:* A cell  $I_j$  is called a good cell, if  $q^m = \{\bar{q}_m, q_{m \mp \frac{1}{2}}^\pm, q_m^l, q_m^r\}$  with  $m = j-1, j, j+1$ , fall into the same linear, convex or concave region of the flux function  $f(q)$ . Otherwise, it is defined to be a nonconvex troubled cell.



4. Update  $q_j^l$  and  $q_j^r$  for a nonconvex troubled cell  $I_j$ .

Perform a first order refined mesh evolution. Specifically, the first order scheme is used to numerically evolve (3.1) with the initial condition

$$\bar{q}_l^n + s_l^n \sigma_l, \text{ for } x \in I_l, \ l = j-1, j, j+1 \quad (3.16)$$

and a periodic boundary condition on  $I_{j-1} \cup I_j \cup I_{j+1}$  for time  $\Delta t$  to obtain  $\tilde{q}^{n+1}|_{I_j}$ . Let  $I_j$  be uniformly discretized by

$$I_j = \cup_{m=1}^N [y_{m-\frac{1}{2}}, y_{m+\frac{1}{2}}], \ \delta x = y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}} = \Delta x/N, \quad (3.17)$$

then  $\tilde{q}^{n+1}|_{I_j}$  is approximated by a piecewise constant function sitting on the refined numerical mesh with the truncation error  $\sim \mathcal{O}(\delta x) = \mathcal{O}(\Delta x^2)$ .  $s_j^{n+1}$  is numerically implemented by finding  $y$  among the refined cell boundaries  $y_{m+\frac{1}{2}} (m = 0, \dots, N)$  that minimized  $\zeta(y)$  as defined in (3.10). Then

$$q_j^{l,n+1} = \overline{\tilde{q}_j^{n+1}} - \frac{1}{2} s_j^{n+1}, \ q_j^{r,n+1} = \overline{\tilde{q}_j^{n+1}} + \frac{1}{2} s_j^{n+1}. \quad (3.18)$$

5. Update the cell averages  $\bar{q}^{n+1}$  and  $\bar{\xi}^{n+1}$  by (3.4).
6. Update  $q_{j \mp \frac{1}{2}}^\pm$  by the HWENO reconstruction procedure.
7. For a good cell  $I_j$ , update  $q_j^l$  and  $q_j^r$  by setting  $q_j^l = u_{j-\frac{1}{2}}^+$  and  $q_j^r = q_{j+\frac{1}{2}}^-$ .

**REMARK 2.** Note that the modification of  $\xi_{j+\frac{1}{2}}^\pm$  is a first order modification on derivative values. Because the first order on derivative values is enough to get a second order scheme.

**REMARK 3.** The implementation of the procedure to find  $s_j^{n+1}$  in (3.18) is computationally expensive. Due to the costly implementation of the high order scheme with the second order entropic projection, we only adopt the first order modification to modify the FV HWENO for 1D system and 2D scalar problems.

### 3.5 Extension to 2D scalar conservation laws

Extensions of the FV HWENO scheme with the first order modification from 1D to 2D cases are straightforward. For instance, we consider a 2D problem on a rectangular domain  $[a, b] \times [c, d]$ :

$$q_t + f(q)_x + g(q)_y = 0. \quad (3.19)$$

Let  $\xi = \frac{\partial q}{\partial x}$ ,  $\eta = \frac{\partial q}{\partial y}$ . Taking the derivatives of (3.19), we obtain

$$\xi_t + \mathcal{H}(q, \xi)_x + \mathcal{R}(q, \xi)_y = 0, \quad (3.20)$$

$$\eta_t + \mathcal{K}(q, \eta)_x + \mathcal{S}(q, \eta)_y = 0, \quad (3.21)$$

where  $\mathcal{H}(q, \xi) = f'(q)\xi$ ,  $\mathcal{R}(q, \xi) = g'(q)\xi$ ,  $\mathcal{K}(q, \eta) = f'(q)\eta$ ,  $\mathcal{S}(q, \eta) = g'(q)\eta$ .

We consider a set of uniform mesh

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x - \frac{1}{2}} < x_{N_x + \frac{1}{2}} = b, \quad \Delta x = \frac{b-a}{N_x},$$

$$c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y - \frac{1}{2}} < y_{N_y + \frac{1}{2}} = d, \quad \Delta y = \frac{d-c}{N_y},$$

with  $I_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ . A semi-discrete FV HWENO discretization is given by

$$\begin{cases} \frac{d}{dt} \bar{q}_{ij} = -\frac{1}{\Delta x} (\hat{f}_{i+\frac{1}{2},j} - \hat{f}_{i-\frac{1}{2},j}) - \frac{1}{\Delta y} (\hat{g}_{i,j+\frac{1}{2}} - \hat{g}_{i,j-\frac{1}{2}}), \\ \frac{d}{dt} \bar{\xi}_{ij} = -\frac{1}{\Delta x} (\hat{\mathcal{H}}_{i+\frac{1}{2},j} - \hat{\mathcal{H}}_{i-\frac{1}{2},j}) - \frac{1}{\Delta y} (\hat{\mathcal{R}}_{i,j+\frac{1}{2}} - \hat{\mathcal{R}}_{i,j-\frac{1}{2}}), \\ \frac{d}{dt} \bar{\eta}_{ij} = -\frac{1}{\Delta x} (\hat{\mathcal{K}}_{i+\frac{1}{2},j} - \hat{\mathcal{K}}_{i-\frac{1}{2},j}) - \frac{1}{\Delta y} (\hat{\mathcal{S}}_{i,j+\frac{1}{2}} - \hat{\mathcal{S}}_{i,j-\frac{1}{2}}), \end{cases} \quad (3.22)$$

where  $\bar{q}_{ij} = \frac{1}{\Delta x \Delta y} \int_{I_{ij}} q dx dy$ ,  $\bar{\xi}_{ij} = \frac{1}{\Delta y} \int_{I_{ij}} \frac{\partial q}{\partial x} dx dy$ ,  $\bar{\eta}_{ij} = \frac{1}{\Delta x} \int_{I_{ij}} \frac{\partial q}{\partial y} dx dy$ . We define  $\hat{f}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(q(x_{i+\frac{1}{2}}, y)) dy$ ,  $\hat{\mathcal{H}}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f'(q(x_{i+\frac{1}{2}}, y)) \xi dy$ ,  $\hat{\mathcal{K}}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f'(q(x_{i+\frac{1}{2}}, y)) \eta dy$  as the average of the fluxes over the right boundary of cell  $I_{ij}$  and  $\hat{g}_{i,j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(q(x, y_{j+\frac{1}{2}})) dx$ ,  $\hat{\mathcal{R}}_{i,j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g'(q(x, y_{j+\frac{1}{2}})) \xi dx$ ,  $\hat{\mathcal{S}}_{i,j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g'(q(x, y_{j+\frac{1}{2}})) \eta dx$  as the average of fluxes over the top boundary of cell. The fluxes  $\hat{f}_{i+\frac{1}{2},j}$ ,  $\hat{\mathcal{H}}_{i+\frac{1}{2},j}$ ,  $\hat{\mathcal{K}}_{i+\frac{1}{2},j}$  are evaluated by applying the  $L$ -points Gauss quadrature rule for integration. For example,

$$\hat{f}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(q(x_{i+\frac{1}{2}}, y)) dy = \sum_{i_g}^L \omega_{i_g} \hat{f}(q_{i+\frac{1}{2},i_g}^-, q_{i+\frac{1}{2},i_g}^+), \quad (3.23)$$

$$\hat{\mathcal{H}}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f'(q(x_{i+\frac{1}{2}}, y)) \xi dy = \sum_{i_g}^L \omega_{i_g} \hat{\mathcal{H}}(q_{i+\frac{1}{2},i_g}^-, q_{i+\frac{1}{2},i_g}^+; \xi_{i+\frac{1}{2},i_g}^-, \xi_{i+\frac{1}{2},i_g}^+), \quad (3.24)$$

with the Lax-Friedrich flux as in (3.5), where  $q_{i+\frac{1}{2},i_g}^\pm$  and  $\xi_{i+\frac{1}{2},i_g}^\pm$  are the approximated values to the left and right limits of  $q(x_{i+\frac{1}{2}}, y_{i_g})$  and  $\xi(x_{i+\frac{1}{2}}, y_{i_g})$  with  $y_{i_g}$  being the Gaussian quadrature points over  $[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$  and  $\omega_{i_g}$  being the corresponding quadrature weight.

The modified high order schemes can be summarized as follows, after a suitable initialization to obtain  $\bar{q}^0$ ,  $\bar{\xi}^0$  and  $\bar{\eta}^0$ .

1. Perform the fourth order HWENO reconstruction. We only illustrate the reconstruction idea of HWENO, and refer to the original paper [24] for details. We relabel the cell  $I_{ij}$  and its neighboring cells as  $I_1, \dots, I_9$  as shown in Figure 3.2, where  $I_{ij}$  is relabeled as  $I_5$ .

7	8	9	j+1
4	5	6	j
1	2	3	j-1
i-1	i	i+1	

Figure 3.2: The big stencil.

We construct the quadratic polynomials  $p_n(x, y)$  ( $n = 1, \dots, 8$ ) in the following stencils,  $S_1 = \{I_1, I_2, I_4, I_5\}$ ,  $S_2 = \{I_2, I_3, I_5, I_6\}$ ,  $S_3 = \{I_4, I_5, I_7, I_8\}$ ,  $S_4 = \{I_5, I_6, I_8, I_9\}$ ,  $S_5 = \{I_1, I_2, I_3, I_4, I_5, I_7\}$ ,  $S_6 = \{I_1, I_2, I_3, I_5, I_6, I_9\}$ ,  $S_7 = \{I_1, I_4, I_5, I_7, I_8, I_9\}$ ,  $S_8 = \{I_3, I_5, I_6, I_7, I_8, I_9\}$ , to approximate  $q(x, y)$ . For instance, a quadratic polynomial can be reconstructed based on the information  $\{\bar{q}_1, \bar{q}_2, \bar{q}_4, \bar{q}_5, \bar{\xi}_4, \bar{\eta}_2\}$  in the stencil  $S_1$ . Such reconstruction will reconstruct a quadratic polynomial on  $I_{ij}$ . Similar reconstructions can be done for stencil  $S_2$ ,  $S_3$  and  $S_4$ . For stencil  $S_5$  to  $S_8$ , only cell averages are used in the reconstruction process. We remark that other combination of information are possible for reconstructing 2D quadratic polynomial. The one we just mentioned seems to be very robust and is implemented in our numerical experiments. If we choose the linear weights denoted by  $\gamma_1^{(l)}, \dots, \gamma_8^{(l)}$  such that

$$q(G_l) = \sum_{n=1}^8 \gamma_n^{(l)} p_n(G_l) \quad (3.25)$$

is valid for any polynomial  $q$  of degree at most 3, leading to a fourth-order approximation of  $q$  at the point  $G_l$  for all sufficiently smooth functions  $q$ . Notice that (3.25) holds for any polynomial  $q$  of degree at most 2 if  $\sum_{n=1}^8 \gamma_n^{(l)} = 1$ . There are four additional constraints on the linear weights  $\gamma_1^{(l)}, \dots, \gamma_8^{(l)}$  so that (3.25) holds for  $q = x^3, x^2y, xy^2$  and  $y^3$ . The rest of free parameters are determined by a least square procedure to minimize  $\sum_{n=1}^8 (\gamma_n^{(l)})^2$ .

As for the derivatives (e.g.  $\xi^-(G_l, t)$ ), a third-order approximation in each stencil is enough to obtain the fourth-order approximation to  $q(x, y)$ . For instance, a cubic polynomial on  $I_5$  can be reconstructed based on the information  $\{\bar{q}_1, \bar{q}_2, \bar{q}_4, \bar{q}_5, \bar{\xi}_1, \bar{\xi}_4, \bar{\xi}_5, \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_5\}$  in the stencil  $S_1$ . Similar reconstructions can

be done for stencil  $S_2$ ,  $S_3$  and  $S_4$ . The information  $\{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4, \bar{\xi}_5, \bar{\xi}_7\}$  in the stencil  $S_5$  is adopted to approximate  $\xi^-(G_l, t)$ . Finally,  $\gamma_l = \frac{1}{8}$  ( $l = 1, \dots, 8$ ) can be chosen. The nonlinear weights of 2D HWENO reconstruction can be designed by following the way of WENO method.

2. Identify the trouble right cell boundary  $x_{i+\frac{1}{2}}$  when  $y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ :

*Criterion III.* If  $q_{i+\frac{1}{2}, i_g}^\pm$ ,  $\bar{q}_{ij}$  and  $\bar{q}_{i+1, j}$  all fall into the same linear, convex or concave region of the flux function  $f(u)$ . Otherwise, it is defined to be a trouble cell boundary. The trouble cell top boundary  $y_{j+\frac{1}{2}}$  when  $x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$  can be identified similarly.

3. At trouble cell boundaries, modify the numerical flux  $\hat{f}_{i+\frac{1}{2}, i_g}$  and  $\hat{g}_{i_g, j+\frac{1}{2}}$  with a discontinuity indicator

$\phi_{ij} = \frac{\beta_{ij}}{\beta_{ij} + \gamma_{ij}}$  where  $\alpha_{ij} = |\bar{u}_{i-1, j} - \bar{u}_{ij}|^2 + \varepsilon$ ,  $\tau_{ij} = |\bar{u}_{i+1, j} - \bar{u}_{i-1, j}|^2 + \varepsilon$ ,  $\beta_{ij} = \frac{\tau_{ij}}{\alpha_{i-1, j}} + \frac{\tau_{ij}}{\alpha_{i+2, j}}$ ,  $\gamma_{ij} = \frac{(u_{max, j} - u_{min, j})^2}{\alpha_{ij}}$ . Here  $\varepsilon$  is a small positive number taken as  $10^{-6}$  in the code, and  $q_{max, j} = \max_i \{\bar{q}_{ij}\}$  and  $q_{min, j} = \min_i \{\bar{q}_{ij}\}$ .

Then Let  $\hat{f}_{i+\frac{1}{2}, i_g} = \hat{f}(q_{i+\frac{1}{2}, i_g}^{m, -}, q_{i+\frac{1}{2}, i_g}^{m, +})$ ,  $\hat{\mathcal{H}}_{i+\frac{1}{2}, i_g} = \hat{\mathcal{H}}(q_{i+\frac{1}{2}, i_g}^{m, -}, q_{i+\frac{1}{2}, i_g}^{m, +}; \xi_{i+\frac{1}{2}, i_g}^{m, -}, \xi_{i+\frac{1}{2}, i_g}^{m, +})$ , and  $\hat{\mathcal{K}}_{i+\frac{1}{2}, i_g} = \hat{\mathcal{K}}(q_{i+\frac{1}{2}, i_g}^{m, -}, q_{i+\frac{1}{2}, i_g}^{m, +}; \eta_{i+\frac{1}{2}, i_g}^{m, -}, \eta_{i+\frac{1}{2}, i_g}^{m, +})$  where

$$\begin{aligned} q_{i+\frac{1}{2}, i_g}^{m, -} &= (1 - \phi_{ij}^2) q_{i+\frac{1}{2}, i_g}^- + \phi_{ij}^2 \bar{q}_{ij}, \quad q_{i+\frac{1}{2}, i_g}^{m, +} = (1 - \phi_{ij}^2) q_{i+\frac{1}{2}, i_g}^+ + \phi_{ij}^2 \bar{q}_{i+1, j} \\ \xi_{i+\frac{1}{2}, i_g}^{m, -} &= (1 - \phi_{ij}^2) \xi_{i+\frac{1}{2}, i_g}^- + \phi_{ij}^2 \bar{\xi}_{ij}, \quad \xi_{i+\frac{1}{2}, i_g}^{m, +} = (1 - \phi_{ij}^2) \xi_{i+\frac{1}{2}, i_g}^+ + \phi_{ij}^2 \bar{\xi}_{i+1, j} \\ \eta_{i+\frac{1}{2}, i_g}^{m, -} &= (1 - \phi_{ij}^2) \eta_{i+\frac{1}{2}, i_g}^- + \phi_{ij}^2 \bar{\eta}_{ij}, \quad \eta_{i+\frac{1}{2}, i_g}^{m, +} = (1 - \phi_{ij}^2) \eta_{i+\frac{1}{2}, i_g}^+ + \phi_{ij}^2 \bar{\eta}_{i+1, j} \end{aligned}$$

4. Evolve the cell average  $\bar{u}_{ij}$  by (3.22).

**REMARK 4.** The same modification can be applied to FV WENO schemes for 2D conservation laws. FV WENO schemes for reconstructing  $q_{i+\frac{1}{2}, i_g}$  on rectangular meshes was described in [14] and the linear weights can be found in [22], where the negative linear weights should be dealt with by the method in [13].

## 4 Numerical Experiments

In Section 4.1, we compare the fifth order FV HWENO schemes (HWENO5) and the fifth order FV WENO schemes (WENO5) with the first order modification (mod1) and the second order entropic projection (mod2) for solving 1D nonconvex conservational laws. We may apply the monotonicity preserving (MP) limiter to the HWENO scheme, denoted as MPHWENO5. We investigate HWENO5 and WENO5 with the first order

modification for 1D Euler equations with the nonconvex EOS in Section 4.2. In Section 4.3, we present the performance of the modified HWENO scheme (HWENO4) for 2D problems. The numerical fluxes used in this paper are the global Lax-Friedrichs flux.

## 4.1 1D scalar problems

**EXAMPLE 1.** *The nonconvex conservation law*

$$q_t + \left(\frac{q^3}{3}\right)_x = 0, \quad q_0(x) = \sin(\pi x). \quad (4.1)$$

We compute the solution up to  $t = 0.2$ . Table 4.1 gives the  $L^1$  and  $L^\infty$  errors and the corresponding orders of accuracy of the regular and modified HWENO5 and WENO5 schemes. We can see that errors of HWENO5 are smaller than those of WENO5 with the same mesh. Very little difference is observed among the regular and two modified HWENO5 and WENO5 schemes.

Table 4.1:  $q_t + \left(\frac{q^3}{3}\right)_x = 0$  with initial condition  $q_0(x) = \sin(\pi x)$  and periodic boundary conditions. The  $L^1$  and  $L^\infty$  errors and the corresponding orders of accuracy for the regular HWENO5 and WENO5, the corresponding two versions of modified schemes at the time  $t = 0.2$ .

N	HWENO5				WENO5			
	$L_1$ error	Order	$L_\infty$ error	Order	$L_1$ error	Order	$L_\infty$ error	Order
	regular							
100	1.68E-05		2.14E-04		4.42E-05		4.61E-04	
200	7.68E-07	4.45	1.69E-05	3.66	2.24E-06	4.30	4.59E-05	3.33
300	1.17E-07	4.64	2.62E-06	4.59	3.48E-07	4.59	7.68E-06	4.41
400	2.96E-08	4.78	6.44E-07	4.88	9.00E-08	4.71	1.96E-06	4.76
500	1.01E-08	4.81	2.12E-07	4.97	3.12E-08	4.75	6.58E-07	4.88
	mod1							
100	1.68E-05		2.14E-04		4.42E-05		4.61E-04	
200	7.68E-07	4.45	1.69E-05	3.66	2.24E-06	4.30	4.59E-05	3.33
300	1.17E-07	4.64	2.62E-06	4.59	3.48E-07	4.59	7.68E-06	4.41
400	2.96E-08	4.78	6.44E-07	4.88	9.00E-08	4.71	1.96E-06	4.76
500	1.01E-08	4.81	2.12E-07	4.97	3.12E-08	4.75	6.58E-07	4.88
	mod2							
100	1.70E-05		2.14E-04		4.42E-05		4.61E-04	
200	7.78E-07	4.45	1.69E-05	3.66	2.24E-06	4.30	4.59E-05	3.33
300	1.17E-07	4.67	2.62E-06	4.59	3.48E-07	4.59	7.68E-06	4.41
400	2.96E-08	4.78	6.44E-07	4.88	9.00E-08	4.70	1.96E-06	4.76
500	1.01E-08	4.81	2.12E-07	4.97	3.12E-08	4.75	6.58E-07	4.88

**EXAMPLE 2.** *Consider (3.1) with the nonconvex flux  $f(q)$  defined by*

$$f(q) = \begin{cases} 1, & \text{if } q < 1.6 \\ \cos(5\pi(q - 1.8)) + 2.0, & \text{if } 1.6 \leq q < 2.0 \\ -\cos(5\pi(q - 2.2)), & \text{if } 2.0 \leq q < 2.4 \\ 1, & \text{if } q \geq 2.4 \end{cases} \quad (4.2)$$

*with the initial condition*

$$q_0(x) = \begin{cases} 1, & \text{for } x < 0 \\ 3, & \text{for } x \geq 0 \end{cases} \quad (4.3)$$

In the left panel of Figure 4.3, the HWENO5 seems to converge to the entropy solution slowly, which might be related to the fact that the reconstruction of the solution at the rarefaction waves comes from neighboring cells and is not a good approximation when the rarefaction wave is surrounded by two shocks at its early stage of development. As shown in Figure 4.3, the numerical solutions of two modified schemes successfully converge to the correct entropy solution.

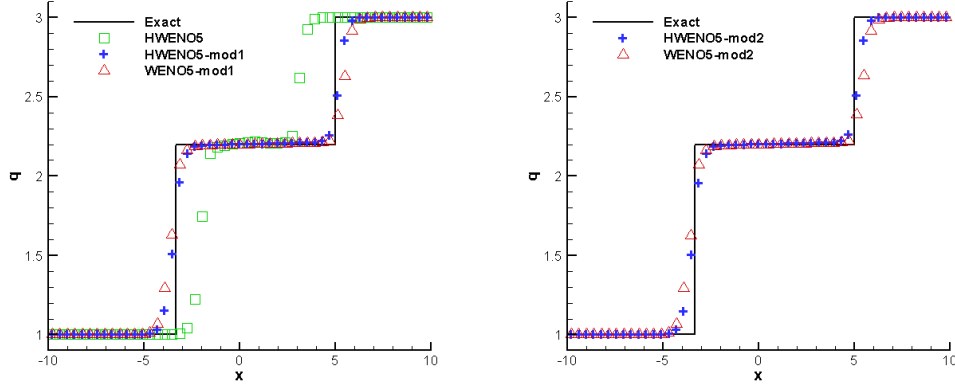


Figure 4.3: Solid line: The exact solutions of the nonconvex scalar conservation law (3.1)-(4.2) with the initial condition (4.3) at the time  $t = 2$ . Left: HWENO5 (squares), HWENO5-mod1 (pluses) and WENO5-mod1 (deltas); Right: HWENO5-mod2 (pluses) and WENO5-mod2 (deltas).  $N = 50$  uniform cells are used.

**EXAMPLE 3.** The nonconvex conservation law (3.1) with the nonconvex flux  $f(q)$  given by (4.2) with the initial condition

$$q_0(x) = \begin{cases} 3, & \text{for } -1 \leq x < 0 \\ 1, & \text{for } 0 \leq x \leq 1 \end{cases} \quad (4.4)$$

and a periodic boundary condition. There is no analytic formula of exact solution for this problem. The reference solution is computed by the Godunov scheme with 400,000 uniform cells. As shown in Fig 4.4, the numerical solution of modified HWENO5 schemes coupled with the monotonicity preserving limiter (MPHWENO5) converge to the correct entropy solution. Note that the solution of HWENO5 (not presented here) will be oscillatory without the MP limiter. The numerical solution of HWENO5 with the first order modification converges to the correct entropy solution with much faster convergence rate, compared to those of WENO5 with the first order modification. Comparable numerical solutions of HWENO5 and WENO5 with the second order modification are observed. We observe faster convergence rate in HWENO5 or WENO5 with the second order modification when compared with those of HWENO5 or WENO5 with the first order modification.

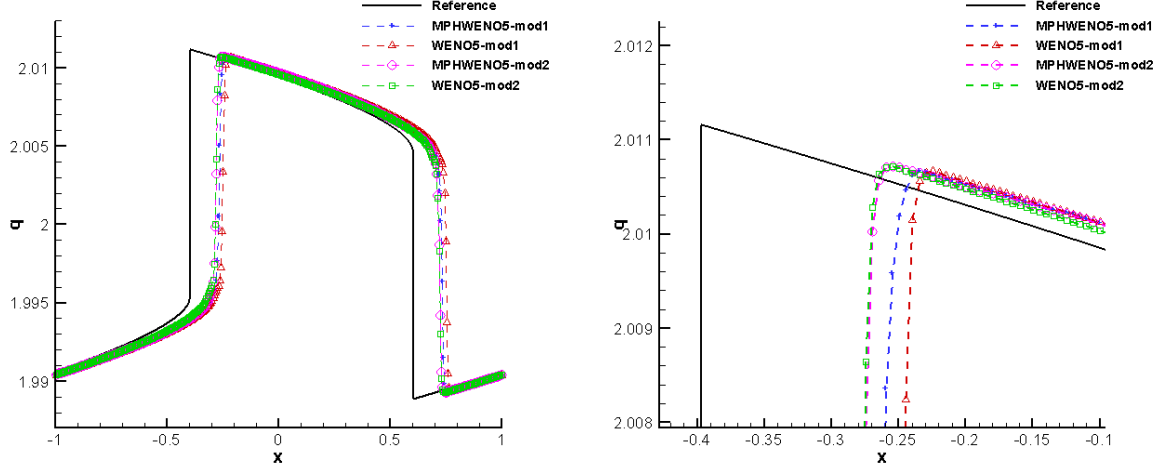


Figure 4.4: Solid lines: The reference solution of (4.4) at the time  $t = 2$ ; Pluses: MPHWENO5 with the first order modification; Deltas: WENO5 with the first order modification; Circles: MPHWENO5 with the second order modification; Squares: WENO5 with the second order modification; The zoom are given in the right;  $N = 400$  uniform cells are used; CFL = 0.01.

**EXAMPLE 4.** Consider the Riemann problem of the nonconvex conservation law (3.1) with the flux function

$$f(q) = \sin(q), \quad (4.5)$$

and the initial condition

$$q_0(x) = \begin{cases} \pi/64, & \text{if } x < 0, \\ 255\pi/64, & \text{if } x \geq 0. \end{cases} \quad (4.6)$$

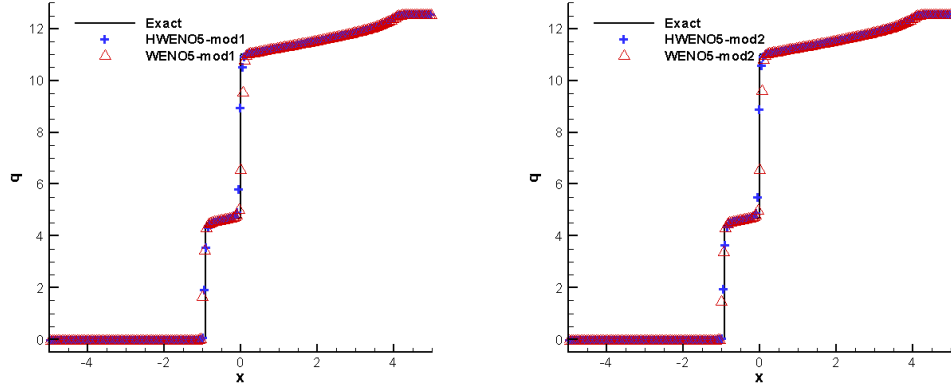


Figure 4.5: Solid lines: the exact solution of (3.1) with  $f(u) = \sin(u)$  and the initial condition (4.6) at the time  $t = 4$ ; HWENO5 (pluses) and WENO5 (deltas) with the first order monotone schemes (left); HWENO5 (pluses) and WENO5 scheme (deltas) with the second order entropy projection (right). using  $N = 200$  uniform cells.

As shown in Fig 4.5, numerical solutions of two modified schemes successfully converge to the correct entropy solution, with the development of a complex solution structure containing a shock, a rarefaction wave, followed by another shock and another rarefaction wave.

## 4.2 1D Euler equations with the nonconvex EOS

We consider the 1D Euler equations of gas dynamics:

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} F(q) = 0, \quad (4.7)$$

where

$$q = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad F(q) = \begin{pmatrix} m \\ \rho u^2 + p \\ u(E + p) \end{pmatrix} \quad (4.8)$$

with the following nonconvex EOS (see [19]):

$$p = p(\rho, e) = ((\gamma - 1)\rho + f(\rho))e, \quad e = \frac{E}{\rho} - \frac{1}{2}u^2, \quad \gamma = 1.4. \quad (4.9)$$

The function  $f$  is given by

$$f(\rho) = \begin{cases} 10e^{\frac{\rho^2}{1-2\rho+0.75\rho^2}}, & \frac{2}{3} < \rho < 2, \\ 0, & \text{otherwise}, \end{cases} \quad (4.10)$$

and  $\rho, u, m = \rho u, p$  and  $E$  are the density velocity, momentum, pressure, and total energy respectively. It can be shown that system (4.2) with the EOS given by (4.9)-(4.10) is hyperbolic. Notice that the system can be rewritten in the nonconservative form:

$$q_t + F'(q)q_x = 0, \quad (4.11)$$

The Jacobian matrix  $F'(q)$  has eigenvalues  $\lambda_1 = u - c$ ,  $\lambda_2 = u$ ,  $\lambda_3 = u + c$ , where  $c$  is the sound speed  $c^2 = p_\rho + \frac{pp_e}{\rho^2}$ . The corresponding right eigenvectors are

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ u - c \\ H - uc \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 + pg'(\rho) \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 1 \\ u + c \\ H + uc \end{pmatrix}, \quad (4.12)$$

where  $g(\rho) = \frac{\rho}{(\gamma-1)\rho+f(\rho)}$  and  $H = \frac{1}{2}u^2 + pg'(\rho) + g(\rho)c^2$ . A matrix  $R(q)$  is formed by the right eigenvectors  $R(q) = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . Letting  $L(q) = R(q)^{-1}$ , then  $L(q)F'(q)R(q) = \Lambda$ , here  $\Lambda$  is the diagonal matrix  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ . Denoting a vector  $\mathbf{l}_k$  to be the  $k$ -th row in  $L(q)$ , then

$$\begin{aligned} \mathbf{l}_1 &= \left( \frac{1}{2} \left( -\frac{2pg'(\rho) - u^2}{2c^2g(\rho)} + \frac{u}{c} \right), -\frac{1}{2} \left( \frac{u}{c^2g(\rho)} + \frac{1}{c} \right), \frac{1}{2c^2g(\rho)} \right), \\ \mathbf{l}_2 &= \left( 1 + \frac{2pg'(\rho) - u^2}{2c^2g(\rho)}, \frac{u}{c^2g(\rho)}, -\frac{1}{c^2g(\rho)} \right), \\ \mathbf{l}_3 &= \left( \frac{1}{2} \left( -\frac{2pg'(\rho) - u^2}{2c^2g(\rho)} - \frac{u}{c} \right), -\frac{1}{2} \left( \frac{u}{c^2g(\rho)} - \frac{1}{c} \right), \frac{1}{2c^2g(\rho)} \right). \end{aligned}$$

### Characteristic-wise FV HWENO for 1D nonconvex system

1. Compute the divided or undivided differences of the cell averages  $\bar{q}$  and  $\bar{\xi}$ , for all  $i$ .



2. For the equations,  $q_{i-\frac{1}{2}}^+, q_{i+\frac{1}{2}}^-, \xi_{i-\frac{1}{2}}^+, \xi_{i+\frac{1}{2}}^-$  are obtained by doing the following:

(a) Compute the right eigenvectors, the left eigenvectors of the Jacobian  $F'(q_i)$  and denote them by

$$R = R(q_i), \quad L = L(q_i).$$

(b) Transform the cell average  $\bar{q}$  and  $\bar{\xi}$  which are in the potential of the HWENO reconstructions for

obtaining  $q_{i-\frac{1}{2}}^+, q_{i+\frac{1}{2}}^-, \xi_{i-\frac{1}{2}}^+, \xi_{i+\frac{1}{2}}^-$ , to the local characteristic fields by using

$$\bar{\mu} = L\bar{q}_j, \quad \bar{\nu} = L\bar{\xi}_j, \quad j \text{ in a neighborhood of } i; \quad (4.13)$$

(c) Perform the scalar HWENO reconstruction for each component of the characteristic variable  $\bar{q}$  and  $\bar{\xi}$

to obtain the corresponding component of the reconstruction  $\mu_{i-\frac{1}{2}}^+, \mu_{i+\frac{1}{2}}^-, \nu_{i-\frac{1}{2}}^+, \nu_{i+\frac{1}{2}}^-$ .

(d) Transform back into physical space by using

$$q_{i-\frac{1}{2}}^+ = R\mu_{i-\frac{1}{2}}^+, \quad q_{i+\frac{1}{2}}^- = R\mu_{i+\frac{1}{2}}^-, \quad \xi_{i-\frac{1}{2}}^+ = R\nu_{i-\frac{1}{2}}^+, \quad \xi_{i+\frac{1}{2}}^- = R\nu_{i+\frac{1}{2}}^-. \quad (4.14)$$

3.  $q_{j+\frac{1}{2}}^\pm$  and  $\xi_{j+\frac{1}{2}}^\pm$  are modified into  $q_{j+\frac{1}{2}}^{m,\pm}$  and  $\xi_{j+\frac{1}{2}}^{m,\pm}$  by the first order modification in Section 3.2. In this example, there are two inflection points  $\rho = 1/0.862$  and  $\rho = 1/1.21$  (see [5]).

4. Apply the Lax-Friedrichs flux (3.5) to compute the fluxes  $\hat{f}$  and  $\hat{\mathcal{H}}$ ; then form the scheme (3.4).

**EXAMPLE 5.** We consider Euler equations (4.2) with EOS (4.9). The domain is  $[-1, 1]$ . The initial condition is set to be  $\rho(x, 0) = 1 + 0.3 \sin(2\pi x)$ ,  $u(x, 0) = 1$ ,  $p(x, 0) = 1$ . The exact solution is  $\rho(x, t) = 1 + 0.3 \sin(2\pi(x - t))$ ,  $u(x, t) = 1$ ,  $p(x, t) = 1$ . This example has two inflection points.

Table 4.2: Euler equations (4.2) with the initial condition  $\rho(x, 0) = 1 + 0.3 \sin(2\pi x)$ ,  $u(x, 0) = 1$ ,  $p(x, 0) = 1$ . The  $L^1$  and  $L^\infty$  errors and the corresponding orders of accuracy for the HWENO5 and WENO5, and the corresponding modified schemes at the time  $t = 1$ .

N	HWENO5				WENO5			
	$L_1$ error	Order	$L_\infty$ error	Order	$L_1$ error	Order	$L_\infty$ error	Order
regular								
40	6.13E-02		1.25E-01		1.44E-02		4.54E-02	
80	1.61E-02	1.92	4.45E-02	1.49	8.13E-03	0.83	3.19E-02	0.51
160	5.77E-03	1.48	1.55E-02	1.52	2.24E-03	1.86	1.52E-02	1.07
320	1.91E-05	8.24	2.60E-04	5.90	7.28E-05	4.95	9.08E-04	4.06
640	4.29E-07	5.48	2.55E-06	6.67	1.28E-06	5.82	6.59E-06	7.11
mod1								
40	7.72E-02		1.72E-01		4.79E-02		1.18E-01	
80	3.24E-02	1.26	1.12E-01	0.62	1.43E-02	1.75	3.85E-02	1.62
160	7.72E-03	2.07	1.88E-02	2.58	2.40E-03	2.57	1.59E-02	1.28
320	5.17E-05	7.22	5.23E-04	5.16	9.65E-05	4.64	1.24E-03	3.68
640	1.17E-06	5.46	1.74E-05	4.91	1.59E-06	5.93	9.87E-06	6.97

We present errors in the  $L^1$  and  $L^\infty$  norm and their numerical orders of accuracy of the density  $\rho$  at  $t = 1$ , on uniform meshes having 40, 80, 160, 320, and 640 elements for the HWENO5 scheme in Table 4.2, in comparison with the results of the WENO5 scheme. There is little difference between regular high order schemes and their modified schemes.

**EXAMPLE 6.** We consider Euler equations (4.2) with EOS (4.9) with the initial condition is

$$(\rho, u, p) = \begin{cases} (0.5, 0.6998, -1) & \text{if } x \leq 0, \\ (0.5, 0.6998, 1) & \text{if } x > 0. \end{cases} \quad (4.15)$$

We calculate the numerical solutions at time  $t = 1$  on a uniform grid with  $\Delta x = \frac{1}{100}$ . The reference solution is given by the FV WENO scheme with the first order modification on a uniform grid,  $\Delta x = \frac{1}{2000}$ . In Figure 4.6, comparable solutions are observed for the FV WENO scheme without modification and that with modification. In the left panel of Figure 4.7, there are some oscillations in the solutions of HWENO5 schemes (left panel). They are alleviated by the monotone preserving limiter (right panel).

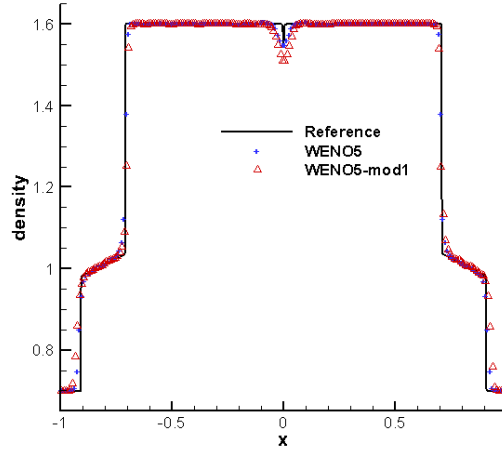


Figure 4.6: Density solved by WENO.  $\Delta x = \frac{1}{100}$ .  $T=1$ .

### 4.3 2D scalar problems

**EXAMPLE 7.** We solve the following nonconvex conservation law in 2D :

$$q_t + \left( \frac{q^3}{3} \right)_x + \left( \frac{q^3}{3} \right)_y = 0,$$

with the initial condition  $q(x, y, 0) = \sin(\pi(x + y)/2)$  and a 4-periodic boundary condition in both directions.

The computational domain for this problem is  $[-2, 2] \times [-2, 2]$ . When  $t = 0.2$  the solution is still smooth.

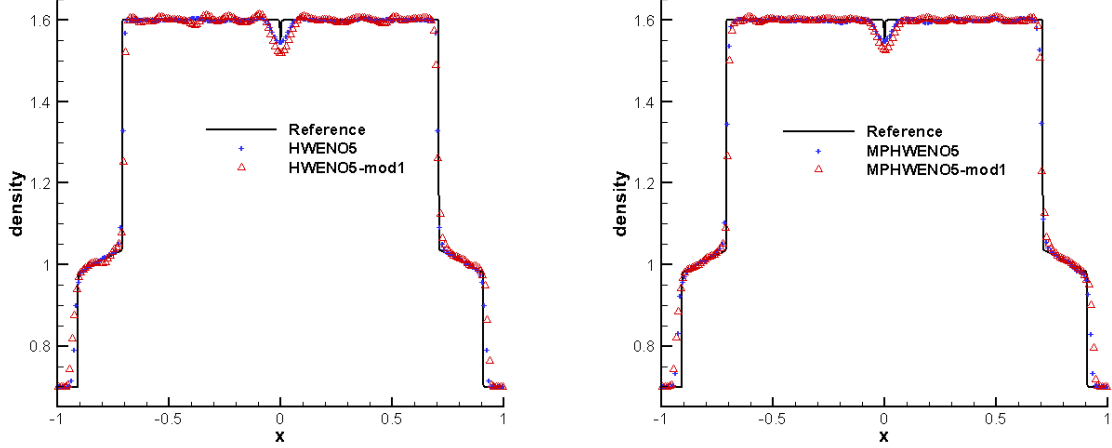


Figure 4.7: Density solved by HWENO and MPHENO.  $\Delta x = \frac{1}{100}$ .  $T=1$ .

Table 4.3:  $q_t + \left(\frac{q^3}{3}\right)_x + \left(\frac{q^3}{3}\right)_y = 0$  with initial condition  $q(x, y, 0) = \sin(\pi(x+y)/2)$  and periodic boundary conditions. The  $L^1$  and  $L^\infty$  errors and the corresponding orders of accuracy for the regular HWENO4 and WENO5, the corresponding modified schemes at the time  $t = 1$ .

N	HWENO4				WENO5			
	$L_1$ error	Order	$L_\infty$ error	Order	$L_1$ error	Order	$L_\infty$ error	Order
regular								
40	3.56E-03	2.36	6.83E-03	0.72	5.26E-04	2.63	4.23E-03	1.08
80	4.25E-04	3.07	1.20E-03	2.51	7.73E-05	2.77	7.86E-04	2.43
160	2.72E-05	3.96	1.15E-04	3.39	5.35E-06	3.85	8.29E-05	3.25
320	1.16E-06	4.56	6.65E-06	4.11	2.31E-07	4.54	4.58E-06	4.18
640	5.68E-08	4.35	3.16E-07	4.39	9.68E-09	4.57	1.39E-07	5.05
mod1								
40	4.26E-03	2.51	6.84E-03	1.56	5.95E-04	3.04	4.28E-03	1.97
80	4.82E-04	3.15	1.20E-03	2.51	7.97E-05	2.90	7.86E-04	2.44
160	3.11E-05	3.95	1.15E-04	3.39	5.42E-06	3.88	8.29E-05	3.25
320	1.38E-06	4.50	6.65E-06	4.11	2.32E-07	4.54	4.58E-06	4.18
640	6.99E-08	4.30	3.16E-07	4.39	9.73E-09	4.58	1.39E-07	5.05

Table 4.3 gives the  $L_1$  errors and the  $L_\infty$  errors and the corresponding orders of the accuracy of the regular and modified FV HWENO scheme and FV WENO scheme.  $\Delta t = \frac{CFL}{\frac{\alpha}{\Delta x} + \frac{\beta}{\Delta y}}$  where  $CFL = 0.4$ ,  $\alpha = \max\{f'(q)\}$ ,  $\beta = \max\{g'(q)\}$ . Expected orders of convergence are observed.

**EXAMPLE 8.** We solve the KPP rotating wave problem,

$$u_t + (\sin(u))_x + (\cos(u))_y = 0$$

with the initial condition

$$u(x, y, 0) = \begin{cases} \frac{14}{4}\pi, & \text{if } \sqrt{x^2 + y^2} \leq 1, \\ \frac{\pi}{4}, & \text{otherwise.} \end{cases}$$

This test was originally proposed in Kurganov et al. [5]. It is challenging to many high-order numerical schemes

because the solution has a two-dimensional composite wave structure.

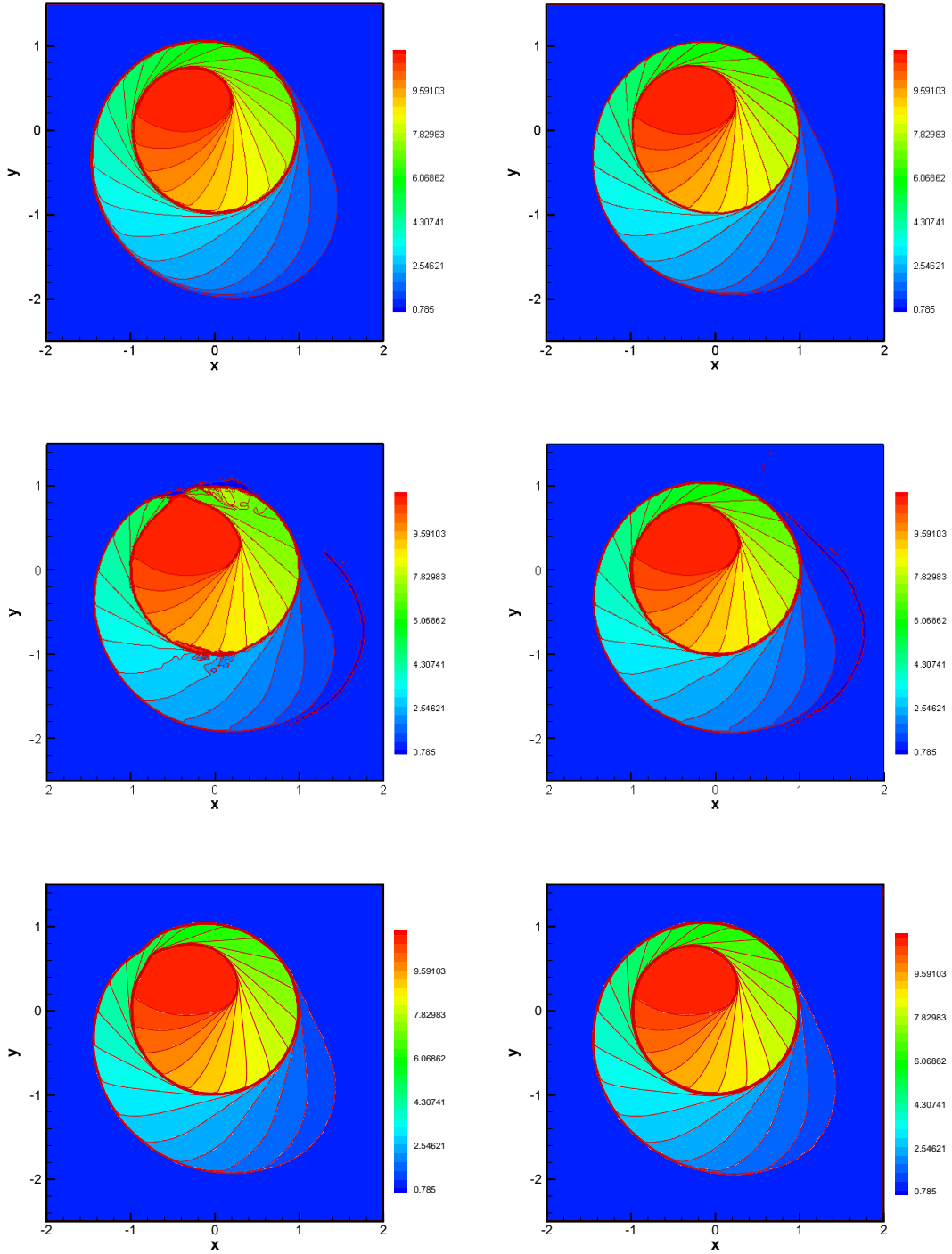


Figure 4.8: KPP problem at time  $t = 1$ . 30 equally spaced solution contours from 0.785 to 11.0. First row: first-order approximation with  $400 \times 400$  cells; first-order approximation with  $1000 \times 1000$  cells. Second row: regular HWENO4 with  $400 \times 400$  cells; modified HWENO4 with  $400 \times 400$  cells. Third row: regular WENO with  $400 \times 400$  cells; modified WENO with  $400 \times 400$  cells.

In Figure 4.8, we show the contours of the solution at  $t = 1$ . In the left panel, it is observed that neither

*HWENO4 or WENO5 schemes can capture composite wave structures. The composite wave structures are well captured by the HWENO4 or WENO5 with the first order modification as shown on the right panel.*

## 5 Concluding remarks

We proposed modified fifth order FV HWENO schemes for nonconvex conservation laws based on the idea of [11], emphasizing convergence to the entropy solutions. The robustness of modified FV HWENO schemes is showed by several representative examples including 1D systems and 2D problems. The comparable solutions are also showed by modified FV WENO schemes.

## References

- [1] F. BOUCHUT, C. BOURDARIAS, AND B. PERTHAME, *A MUSCL method satisfying all the numerical entropy inequalities*, Mathematics of Computation of the American Mathematical Society, 65 (1996), pp. 1439–1461.
- [2] B. COCKBURN AND C.-W. SHU, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework*, Mathematics of Computation, 52 (1989), pp. 411–435.
- [3] M. G. CRANDALL AND A. MAJDA, *Monotone difference approximations for scalar conservation laws*, Mathematics of Computation, 34 (1980), pp. 1–21.
- [4] J.-L. GUERMOND, R. PASQUETTI, AND B. POPOV, *Entropy viscosity method for nonlinear conservation laws*, Journal of Computational Physics, 230 (2011), pp. 4248–4267.
- [5] A. KURGANOV, G. PETROVA, AND B. POPOV, *Adaptive semidiscrete central-upwind schemes for nonconvex hyperbolic conservation laws*, SIAM Journal on Scientific Computing, 29 (2007), pp. 2381–2401.
- [6] R. MENIKOFF AND B. J. PLOHR, *The Riemann problem for fluid flow of real materials*, Reviews of modern physics, 61 (1989), p. 75.
- [7] S. MÜLLER AND A. VOSS, *The Riemann problem for the Euler equations with nonconvex and nonsmooth equation of state: construction of wave curves*, SIAM Journal on Scientific Computing, 28 (2006), pp. 651–681.

- [8] J. QIU AND C.-W. SHU, *Hermite WENO schemes and their application as limiters for Runge–Kutta discontinuous Galerkin method: one-dimensional case*, Journal of Computational Physics, 193 (2004), pp. 115–135.
- [9] ———, *Hermite WENO schemes and their application as limiters for Runge–Kutta discontinuous Galerkin method II: Two dimensional case*, Computers & Fluids, 34 (2005), pp. 642–663.
- [10] ———, *Hermite WENO schemes for Hamilton–Jacobi equations*, Journal of Computational Physics, 204 (2005), pp. 82–99.
- [11] J.-M. QIU AND C.-W. SHU, *Convergence of high order finite volume weighted essentially non-oscillatory scheme and discontinuous Galerkin method for nonconvex conservation laws*, SIAM Journal on Scientific Computing, 31 (2008), p. 584.
- [12] S. SERNA AND A. MARQUINA, *Anomalous wave structure in magnetized materials described by non-convex equations of state*, Physics of Fluids (1994–present), 26 (2014), p. 016101.
- [13] J. SHI, C. HU, AND C.-W. SHU, *A technique of treating negative weights in WENO schemes*, Journal of Computational Physics, 175 (2002), pp. 108–127.
- [14] C.-W. SHU, *High order weighted essentially non-oscillatory schemes for convection dominated problems*, SIAM Review, 51 (2009), pp. 82–126.
- [15] C.-W. SHU AND S. OSHER, *Efficient implementation of essentially non-oscillatory shock-capturing schemes*, Journal of Computational Physics, 77 (1988), pp. 439–471.
- [16] A. SURESH AND H. HUYNH, *Accurate monotonicity-preserving schemes with Runge–Kutta time stepping*, Journal of Computational Physics, 136 (1997), pp. 83–99.
- [17] Z. TAO, F. LI, AND J. QIU, *High-order central Hermite WENO schemes on staggered meshes for hyperbolic conservation laws*, Journal of Computational Physics, 281 (2015), pp. 148–176.
- [18] B. VAN LEER, *Towards the ultimate conservative difference scheme. IV. a new approach to numerical convection*, Journal of Computational Physics, 23 (1977), pp. 276–299.
- [19] B. WANG AND H. GLAZ, *Second order Godunov-like schemes for gas dynamics with a nonconvex equation of state*, in AIAA Computational Fluid Dynamics Conference, 14 th, Norfolk, VA, 1999.

- [20] Z. XU AND C.-W. SHU, *Anti-diffusive finite difference WENO methods for shallow water with transport of pollutant*, Journal of Computational Mathematics, 24 (2006), pp. 239–251.
- [21] H. YANG, *Convergence of Godunov type schemes*, Applied Mathematics Letters, 9 (1996), pp. 63–67.
- [22] X. ZHANG AND C.-W. SHU, *On maximum-principle-satisfying high order schemes for scalar conservation laws*, Journal of Computational Physics, 229 (2010), pp. 3091–3120.
- [23] F. ZHENG AND J. QIU, *Directly solving the Hamilton–Jacobi equations by Hermite WENO Schemes*, Journal of Computational Physics, 307 (2016), pp. 423–445.
- [24] J. ZHU AND J. QIU, *A class of the fourth order finite volume Hermite weighted essentially non-oscillatory schemes*, Science in China Series A: Mathematics, 51 (2008), pp. 1549–1560.